Coherent Sequencing of Early Mathematics Content for Students with Autism

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Tech Connection

Sites.google.com/pattan.net/ptnmath
PaTTAN’s Mission

The mission of the Pennsylvania Training and Technical Assistance Network (PaTTAN) is to support the efforts and initiatives of the Bureau of Special Education, and to build the capacity of local educational agencies to serve students who receive special education services.

PDE’s Commitment to Least Restrictive Environment (LRE)

Our goal for each child is to ensure Individualized Education Program (IEP) teams begin with the general education setting with the use of Supplementary Aids and Services before considering a more restrictive environment.
Session Description

Foundational numeracy concepts are taken for granted in education. It is often assumed that students will possess certain skills before they even begin formal education in mathematics.

This assumption can create gaps in learning and lead to remediation, instead of altering the original instructional sequence to be more coherent.

Students with Autism often have delays in language acquisition, which leads to delayed instruction in mathematics. This delay in mathematics learning presents educators with a unique opportunity to redefine how we think about early numeracy concepts and design more coherent sequences in mathematics curricula.

Thinking differently about early numeracy?

- Identify skills
- Order skills logically
- Find associated prerequisites
- Teach to mastery/fluency/ across exemplars/ etc…
Session Outline

1. ABA Stuff
2. Early Numeracy Sequencing
3. Counting Principles
4. Operations
Pop Quiz!

Math is a ____________ Language

5 Strands of Mathematical Proficiency

Prerequisites  Conceptual Understanding  Math Topic

Strategic Competence  Productive Disposition
Adaptive Reasoning  Procedural Fluency

(NRC, 2001)
What is **conceptual understanding?**

**Extended Tacts**

- **Generalization** must occur
  - Can apply to *novel items* without explicit teaching
  - Across...
    1. People
    2. Places
    3. Materials
    4. Instructions
    5. Time

- **Feature/Function/Class**
  - Tacting *critical features* may facilitate concept acquisition

- The tact is involved in the process of *joint control* which assists students in effective *verbal recall* and effective *listener responding.*

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**Atomic Repertoires**

- New combination of skills applied to new behaviors
- Most of our spoken language is a result of ARs

**What are the prerequisite skills needed for the atomic repertoires for the math content?**

- Imitation
- Echoic
- Tacts
- Textual Behavior (reading texts/symbols)
- Transcriptive Behavior (copying text/symbols)
- Etc...

**We must identify the skills in relation to content!**
### From this point on…

I am going to **simplify the ABA Vocabulary** so we can focus on the math.

You can still make connection/improvements if you have that level of background.
Early Numeracy

NCII: Teaching Counting

Teaching Counting:
Considerations for Instruction

Purpose and Overview of Guide

The purpose of this guide is to provide strategies and materials for developing and implementing lessons for students who need intensive instruction in the areas of place value, numeracy, and counting. Resource teachers, special education, Tier Three, and others working with struggling students may find this guide helpful.

Within the Common Core State Standards, place value is taught in the Counting and Cardinality domain in Kindergarten and the Operations and Algebraic Thinking domain in Grades 1 and 2. This grade may be used as these concepts are introduced, or with students in higher grade levels who continue to struggle with the concepts.

The guide is divided into four sections:
1. Sequence of skills as defined by the Common Core State Standards
2. A list of important vocabulary and symbols
3. A brief explanation of the instructional ideas
4. Background

5 min.
Early Numeracy

3 Broad Outcomes
• Conceptual Understanding
• Computational Fluency
• Problem Solving

2 Central Themes
• Place Value
• Basic Arithmetic Operations

Conceptual Understanding (Willingham 2009)
• Understanding meaning and rationale
• Logical, justifiable, knowing the “why”

Computational Fluency (NCTM 2000)
• Efficient, accurate methods to compute
• Accuracy, flexibility, understanding

Problem Solving (Schoenfeld 1992)
• Routine excersizes
• Reaching goal not immediately attainable, “novel”
Early Numeracy: Central Themes

**Place Value (Base 10)**
- Single Digits
- Groups of ten
- Positional Base System

**Basic Arithmetic Operations**
- Addition/Subtraction
- Multiplication/Division

**Inverse Operations**

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**Basic Principles of Counting**

3. **One-to-one** – Counting one “thing” at a time; transfer from uncounted group to counted group (1:1 Correspondance)

1. **Cardinal** – The last count represents the quantity in the counted group (Cardinality)

2. **Stable-order** – Establishes consistent sequence

   - Abstraction – applying counting to like objects, actions, sounds, etc...

4/5. **Order-irrelevance** – Can count in any order
Developmental Dyscalculia

**Number Module Deficit**

**Quantity Meaning & Symbolic Meaning must be emphasized!**

**Symbolic Access Deficit**

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From Quantity to Computation

1. Understand Individual Quantities (Cardinality)
2. Establish Consistent Count Sequence
3. Apply 1-1 Correspondence

- **4a**: Establish Order Irrelevance
- **4b**: Work towards abstraction

#### Addition
- **5a**: Count all
- **5b**: Take away

#### Subtraction
- **6a**: Count on
- **6b**: Make 5
- **6c**: Make 10
- **7a**: Think Addition
- **7b**: Across 5
- **7c**: Across 10

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**Word Problems**

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# Cardinality

“what numbers represent”

“What does three really mean? What is three-ness”

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## What does “3” really mean?

<table>
<thead>
<tr>
<th>3</th>
<th>three</th>
<th>“three”</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>&quot;1 ... 2 ... 3!&quot;</td>
<td></td>
</tr>
<tr>
<td></td>
<td>“one more than 2”</td>
<td>“one less than 4”</td>
</tr>
<tr>
<td></td>
<td>“is between…”</td>
<td>“is more than…”</td>
</tr>
<tr>
<td></td>
<td>“is the same as…”</td>
<td>“is less than…”</td>
</tr>
</tbody>
</table>

3 units
**Cardinality:** the size of a set

- The number of elements in a set.

“A set of numbers, called $S$, contains the numbers 1, 3, 5, 7, and 9. The cardinality of the set $S$ is 5.”

**If $S = \{1, 3, 5, 7, 9\}$ then $|S| = 5$**

Cardinality begins by learning quantities/patterns. 
Cardinality is enhanced with 1:1 Correspondence. 
- Through 1:1 “counting”
- Through 1:1 “matching”
**Cardinality:** the size of a set

- The number of elements in a set.

“A set of dots, called D, contains the dots \( \bullet, \circ , \odot, \) and \( \bigstar \). The cardinality of the set D is 4.”

\[
\text{if } D = \{ \bullet, \circ , \odot, \bigstar \} \text{ then } |D| = 4
\]

---

**Subitization**

The ability to see a quantity and know how many, without “counting.”

**Perceptual** and **Conceptual**
Subitization

Research indicated that dice patterns and rectangular arrays are the easiest for students to learn.

Don’t go crazy!


Subitizing – “How Many?”
Subitizing – “How Many?”

Connecting Representations of Numbers
Subitization

Verbal Conditional Discrimination must be established.

- What is it?
- What part is it?
- How many?

This is complex verbal behavior.
**Subitization – Tacting a Feature**

<table>
<thead>
<tr>
<th>Trial</th>
<th>Teacher</th>
<th>Learner</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tact Prompt for Part</td>
<td>Presents item</td>
<td>“Six”</td>
</tr>
<tr>
<td></td>
<td>“How many? Six.”</td>
<td></td>
</tr>
<tr>
<td>Tact Transfer</td>
<td>“How many?***”</td>
<td>“Six”</td>
</tr>
<tr>
<td>Distractor(s)</td>
<td>?</td>
<td>?</td>
</tr>
<tr>
<td>Tact Trial Item</td>
<td>Presents item</td>
<td>“Red-veined Dropwing Dragonflies”</td>
</tr>
<tr>
<td></td>
<td>“What are these?”</td>
<td></td>
</tr>
<tr>
<td>Tact Part Check</td>
<td>Presents item</td>
<td>“Six”</td>
</tr>
<tr>
<td></td>
<td>“How many?”</td>
<td></td>
</tr>
</tbody>
</table>

**Error Correction** – Run a contrast correction as part of the distract trial sequence

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**Subitization – Data Collection**

Skills Tracking Sheet

<table>
<thead>
<tr>
<th>Task</th>
<th>Target</th>
<th>Date Introduced</th>
<th>Date Shown</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>shoe</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>car</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>red-veined dragonfly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>plant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>frog</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>snake</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>bus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>car</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>red-veined dragonfly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>plant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>frog</td>
<td></td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>snake</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>bus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>car</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>red-veined dragonfly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>plant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>frog</td>
<td></td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>snake</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>bus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>car</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>red-veined dragonfly</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>plant</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>frog</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>snake</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>bus</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>car</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Subitization – Tacting a Feature

Generalization & discrimination should be present for the items in the set.

The concept of quantity has been developed when the individual can subitize (tact) novel items in a set without explicit training.

Cardinality: the size of a set

four reindeer
4 pumpkins
4

“Four!”
**Cardinality:** the size of a set

This is a setting to ensure the following abilities are generalized across multiple exemplars

- **Words** (saying, writing)
- **Patterns** (identifying, building)
- **Digits** (matching, writing)
- **Assigning units**

**Understanding Individual Quantities**
Potential Prerequisites?

What prerequisite skills might students need to learn about cardinality?

Play time!

Match Game
Go Fish!
War!

Other Ideas?
Stable-Order

“consistent count sequence”
Stable-Order: Consisent Count Sequence

- Establish a consistent count sequence.

Once student understand a set of quantities, those are arranged in order of magnitude to establish a count sequence.

Magnitude – Comparing to other sets
Single Comparison - Greater than, less than, equal
Multiple comparisons - Ordering sets

Single Comparision

- Comparing one number to another

Which is more/less?  
2 or 8
# or #  
4 or 5

More bees or trees?
Single Comparison

- Comparing one number to another

[Diagrams of comparing one number to another]
Single Comparision

- Comparing one number to another

\[ \# \quad \text{is} \quad \begin{array}{c} \text{greater} \\ \text{equal} \\ \text{less} \end{array} \quad \text{than/to} \quad \# \]

The number of bees is ____ than the number of trees.

Multiple Comparisions (ordering)

- Comparing more than one number

Least, greatest, minimum, maximum, middle…

ORDERING

1 2 3 4 5
Multiple Comparisons (ordering)

- Comparing more than one number

Least, greatest, minimum, maximum, middle…

ORDERING

1 2 3 4 5

Stable-Order: Consistent Count Sequence

0 1 3 9 4 10 7 2 5 8 6

"Zero, one, two, three, four, five, six, ..."
Establist a Consistant Count Sequence

Potential Prerequisites?

What prerequisite skills might students need to establish a consistent count sequence?
1:1 Correspondance

“not just counting words”

From Quantity to Computation

- Understand Individual Quantities (Cardinality)
- Establish a Consistent Count Sequence
- Apply 1-1 Correspondence
- Establish Order Irrelevance
- Work towards abstraction
- From Quantity to Computation
- Addition: Count all, Count on, Make 5, Make 10
- Subtraction: Take away, Think Addition, Across 5, Across 10
- Place Value
- Benchmark Numbers
- Word Problems
1:1 Correspondence

- Pairing between two sets, each object in A with one and only one object in B

if \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{a, b, c, d, e\} \)
then \( A \) and \( B \) are in one-to-one correspondance

\((1, a), (2, b), (3, c), (4, d), (5, e)\)

1:1 Correspondence & Cardinality

Cardinality is enhanced through 1:1 Correspondence

Magnitude

More bees or trees?
1:1 Correspondence & Cardinality

Cardinality is enhance through 1:1 Correspondence…

Magnitude

More bees or trees?

1:1 Correspondence & Cardinality

Cardinality is enhance through 1:1 Correspondence…

Magnitude
1: 1 Correspondence

- Pairing between two sets, each object in A with one and only one object in B

\[
\text{if } A = \{ \text{bear}, \text{bear}, \text{bear}, \ldots \} \quad \text{and} \\
B = \{"one", "two", "three", \ldots \}
\]

then ...

Applying the Count Sequence

- **Partitioning** – Moving from “uncounted pile” to “counted pile” – touching?

- **Tagging** – assigning label; placing in “labeled” spot or attaching label
Applying the Count Sequence

<table>
<thead>
<tr>
<th></th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
</table>

0 1 2 3 4 5 6 7 8

Applying the Count Sequence

1 4 5
2 3 6

10
Applying the Count Sequence

“Get four cars.”
Defined set

“How many yellow?”
Discrimination

1 2 3 4 5
6 7 8 9 10

Get four cars.
Defined set

How many yellow?
Discrimination
Applying the Count Sequence

3 + 2 = ___

Apply 1:1 Correspondence

1. Given a set of movable objects (arranged or scattered), count the objects moving them out of the set as counted
2. Given a set of pictured objects (arranged or scattered), count the objects touching them as counted
3. Given a set of objects and a stated quantity, count out the stated subset
4. Given several pictures of objects and a stated quantity, select the set that matches the given quantity

1.1 Given two sets of objects, pair objects and state whether the sets are equivalent
1.2 Given two sets of objects, pair objects and state which sets is larger
1.3 Given two sets of objects, pair objects and state which sets is less
Potential Prerequisites?

What prerequisite skills might students need to learn about 1:1 Correspondance?

Order Irrelevance
Abstraction

“what are we counting”
From Quantity to Computation

Order Irrelevance

• Counting a set of items in different orders always results in the same size set
Abstraction

• Count things that are “not easily re-counted”

Potential Prerequisites?

What prerequisite skills might students need to learn about flexibly and abstractly?
Operations

From Quantity to Computation

1. Understand Individual Quantities (Cardinality)
2. Establish a Consistent Count Sequence
3. Apply 1-to-1 Correspondence
4a. Establish Order Irrelevance
4b. Work towards abstraction
5a. Establish Benchmark Numbers
5b. Benchmark Numbers
6a. Benchmark Numbers
6b. Benchmark Numbers

Addition
5a. Count all
5b. Count on
6a. Make 5
6b. Make 10

Subtraction
5a. Take away
5b. Think Addition
6a. Across 5
6b. Across 10

Place Value

Word Problems

76
The Mathematics Framework, Appendix F

5 min.

Appendix F: Methods used for solving single-digit addition and subtraction problems.

Level 1: Direct Modeling by Counting All or Taking Away.
- Represent situations or numerical problems with groups of objects, a drawing, or欧阳.
- Model the situations by composing two added groups or decomposing a single group.
- Adding (3 + 5 = 8): Represent each addend for a group of objects. Put the two groups together. Close the total. Use the strategy for kids to find 10.

Level 2: Count on
- Subtracting (14 - 5 = 9): Removes the world by a group of objects. Ask the known addend number of objects away. Count the remaining group of objects to find the unknown addend. Use the strategy for kids to find 10.

Levels | Count All | Take Away
--- | --- | ---
Level 1: Count on
Level 2: Count on
Level 3: Decompose
- Make a ten (general): one addend breaks apart to make 10 with the other addend
- Make a ten (from 5's within each addend)

Doubles + n
- 5 + 8
- 12 + 2 = 14

Note: Many children attempt to count down for subtraction, but counting down is difficult and error-prone. Children are much more successful with counting on, but it makes subtraction as easy as addition.
Early Addition/Subtraction

“practice counting with symbols”

From Quantity to Computation

1. Understand Individual Quantities (Cardinality)
2. Establish a Consistent Count Sequence
3. Apply 1-1 Correspondence
4a. Establish Order Irrelevance
4b. Work towards abstraction

5a. Count all
5b. Take away
6a. Cover all
6b. Think Addition
7a. Count on
7b. Across 5
8a. Make 5
8b. Across 10
9a. Benchmark Numbers
9b. Benchmark Numbers

Place Value

Word Problems
Count All – Take Away

\[ 4 + 3 = 7 \]
\[ (1 + 1 + 1 + 1) + (1 + 1 + 1) = \]

[Diagram with red circles arranged in a grid]

Count All – Take Away

\[ 4 + 3 = 7 \]
\[ (1 + 1 + 1 + 1) + (1 + 1 + 1) = \]

[Diagram with hand icons and a number line]
Count All – Take Away

\[ 5 - 3 = 2 \]
\[(1 + 1 + 1 + 1 + 1) - (1 + 1 + 1) = \]
\[1 + 1 + \color{red} \times + \color{red} \times + \color{red} \times + \color{red} 1) \rightarrow \color{red} \times \rightarrow \color{red} \times \rightarrow \color{red} \times = \]

Count All – Take Away

\[ 5 - 3 = 2 \]
\[(1 + 1 + 1 + 1 + 1) - (1 + 1 + 1) = \]
\[1 + 1 + \color{red} \times + \color{red} \times + \color{red} \times + \color{red} \times \rightarrow \color{red} \times \rightarrow \color{red} \times \rightarrow \color{red} \times = \]
From Quantity to Computation

Count On – Think Addition

\[ 6 + 2 = 8 \]
\[ 6 + (1 + 1) = \]

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{count_on.png}
\end{figure}
Count On – Think Addition

6 + 2 = 8
6 + (1 + 1) =

Count On – Think Addition

6 − 2 = 4
6 − (1 + 1) = 6 − 1 − 1 =

Count Back
Count On – Think Addition

6 − 2 = 4
2 + 4 = 6

Count On – Think Addition

6 − 2 = 4
2 + 4 = 6
From Quantity to Computation

Word Problems

Make 5 – Across 5

\[ 4 + 4 = 8 \]
\[ 4 + (1 + 3) = (4 + 1) + 3 = 5 + 3 = \]
Make 5 – Across 5

\[
\begin{align*}
4 + 4 &= 8 \\
4 + (1 + 3) &= \\
(4 + 1) + 3 &= 5 + 3 = \\
\end{align*}
\]

Make 5 – Across 5

\[
\begin{align*}
7 - 4 &= 3 \\
7 - (2 + 2) &= \\
(7 - 2) - 2 &= 5 - 2 = \\
\end{align*}
\]
Make 5 – Across 5

7 - 4 = 3
7 - (2 + 2) =
7 - 2 - 2 = 5 - 2 =

Make 10 – Across 10

8 + 6 = 14
8 + (2 + 4) =
(8 + 2) + 4 = 10 + 4 =
Make 10 – Across 10

8 + 6 = 14
8 + (2 + 4) =
(8 + 2) + 4 = 10 + 4 =

Make 10 – Across 10

9 + 9 = 18
9 + (1 + 8) =
(9 + 1) + 8 = 10 + 8 =
Make 10 – Across 10

\[ 9 + 9 = 18 \]
\[ 9 + (1 + 8) = \]
\[ (9 + 1) + 8 = 10 + 8 = \]

Make 10 – Across 10

\[ 16 - 9 = 7 \]
\[ 16 - (6 + 3) = \]
\[ (16 - 6) - 3 = 10 - 3 = \]
Make 10 – Across 10

16 – 9 = 7
16 – (6 + 3) =
16 – 6 – 3 = 10 – 3 =

From Quantity to Computation

1. Understand Individual Quantities (Cardinality)
2. Establish a Consistent Count Sequence
3. Apply 1-1 Correspondence
4a. Establish Order Irrelevance
4b. Work towards abstraction
5a. Establish Place Value
5b. Benchmark Numbers
6a. Benchmark Numbers
6b. Benchmark Numbers

Word Problems

Addition
5a. Count all
6a. Count on
7a. Make 5
8a. Make 10

Subtraction
5b. Take away
6b. Think Addition
7b. Across 5
8b. Across 10

Place Value
Importance of “tens”

• Mental Math is handled in chunks, not an algorithm
• Flexibility with numbers – **NUMBER SENSE**
• Leads to place value

What is **Number Sense**?

“a child’s fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons”

(Gersten & Chard, 1999)
Importance of “tens”

• Mental Math is handled in chunks, not an algorithm
• Flexibility with numbers – **NUMBER SENSE**
• Leads to place value

\[
\begin{align*}
34 & \quad (34 - 6) - 10 \\
- 16 & \quad (34 - 4 - 2) - 10 \\
\end{align*}
\]

\[
\begin{align*}
28 - 10 & \quad 18
\end{align*}
\]

Place Value

“seeing sets of 10”
From Quantity to Computation

1. Understand Individual Quantities (Cardinality)
2. Establish a Consistent Count Sequence
3. Apply 1-1 Correspondence
4a. Establish Order Irrelevance
4b. Work towards abstraction
5a Benchmark Numbers
5b Benchmark Numbers
6a Benchmark Numbers

Addition

5b Count all
5a Count on
6b Make 5
6a Make 10
7b Across 5
7a Across 10
8b Across 10
8a Take away
6b Think Addition

Subtraction

Place Value

Ten-Frame Progression

1 2 7
+ 3 6
6 3

6 60 + 3

10 11
From Quantity to Computation

1. Understand individual quantities (curiosity)
2. Establish a consistent count sequence
3. Apply 1-1 correspondence
4a. Establish order irrelevance
4b. Work towards abstraction
5a. Count all
5b. Take away
6a. Count on
6b. Think addition
7a. Make 5
7b. Across 5
8a. Make 10
8b. Across 10

Word Problems

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